Density Waves in the Calogero Model - Revisited

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Abstract

The Calogero model bears, in the continuum limit, collective excitations in the form of density waves and solitary modulations of the density of particles. This sector of the spectrum of the model was investigated, mostly within the framework of collective field theory, by several authors, over the past fifteen years or so. In this work we shall concentrate on periodic solutions of the collective BPS-equation (also known as "finite amplitude density waves"), as well as on periodic solutions of the full static variational equations which vanish periodically (also known as "large amplitude density waves"). While these solutions are not new, we feel that our analysis and presentation add to the existing literature, as we explain in the text. In addition, we show that these solutions also occur in a certain two-family generalization of the Calogero model, at special points in parameter space. A compendium of useful identities associated with Hilbert transforms, including our own proofs of these identities, appears in Appendix A. In Appendix B we also elucidate in the present paper some fine points having to do with manipulating Hilbert transforms, which appear ubiquitously in the collective field formalism. Finally, in order to make this paper self-contained, we briefly summarize in Appendix C basic facts about the collective field formulation of the Calogero model.

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1   Introduction

The Calogero Model (CM) [1] - [3] is a well-known exactly solvable many-body system, both at the classical and quantum levels. It describes N particles (considered indistinguishable at the quantum level) on the line, which interact through an inverse-square two-body interaction. Its quantum Hamiltonian is

$$H = -\frac{1}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i{}^2} + \frac{\lambda(\lambda-1)}{2m} \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2} \, , \tag{1}$$

where m is the particles' mass, and the dimensionless coupling constant λ parametrizes the inverse-square interaction between pairs of particles. [1]

The CM and its various descendants continue to draw considerable interest due to their many diverse physical applications. A partial list of these applications can be found, for example, in the introductory section of [4]. For recent reviews on the Calogero- and related models see, e.g., [5, 6]. In addition, for a recent review on the collective-field and other continuum approaches to the spin-Calogero-Sutherland model, see [7].

In the present paper we concentrate on the thermodynamic limit of the CM. In this limit the system is amenable to large-N collective-field formulation [8, 9, 10]. As is well-known, the collective theory offers a continuum field-theoretic framework for studying interesting aspects of many-particle systems. Clearly, a description of the particle systems in terms of continuous fields becomes an effectively good one in the high density limit. In this limit the mean interparticle distance is much smaller

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[1]Note that we did not include in (1) a confining potential. This is not really a problem, as we can always add a very shallow confining potential to regulate the problem (in the case of purely repulsive interactions), or else, consider the particles confined to a very large circle (i.e., consider (1) as the large radius limit of the Calogero-Sutherland model [2]). We shall henceforth tacitly assume that the system is thus properly regularized at large distances.

than any relevant physical length-scale, and the δ-function spikes in the density field (4) below can be smoothed-out into a well-behaved countinuum field. All this is in direct analogy to the hydrodynamical effective description of fluids, which replaces the microscopic atomistic formulation. Of course, the large density limit means that we have taken the large-N limit, as was mentioned above.

The collective-field Hamiltonian for the CM (1) is given by [11]

$$H_{coll} = \frac{1}{2m} \int dx\, \partial_x \pi(x)\, \rho(x)\, \partial_x \pi(x) + \frac{1}{2m} \int dx\, \rho(x) \left(\frac{\lambda-1}{2} \frac{\partial_x \rho}{\rho} + \lambda \!\!\int \frac{dy \rho(y)}{x-y} \right)^2 + H_{sing}\,, \tag{2}$$

where H_{sing} denotes a singular contribution [12]

$$H_{sing} = -\frac{\lambda}{2m} \int dx\, \rho(x)\, \partial_x \left. \frac{P}{x-y} \right|_{y=x} - \frac{\lambda-1}{4m} \int dx\, \partial_x{}^2\, \delta(x-y)|_{y=x}\,, \tag{3}$$

and P is the principal part symbol.

Here,

$$\rho(x) = \sum_{i=1}^{N} \delta(x-x_i) \tag{4}$$

is the collective - or density - field, and

$$\pi(x) = -i \frac{\delta}{\delta \rho(x)} \tag{5}$$

is its canonically conjugate momentum. It follows trivially from (4) that the collective field is a positive operator

$$\rho(x) \geq 0\,, \tag{6}$$

and that it obeys the normalization condition

$$\int\limits_{-\infty}^{\infty} dx\, \rho(x) = N\,. \tag{7}$$

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The latter constraint is implemented by adding to (2) a term $\mu \left(\int\limits_{-\infty}^{\infty} dx\, \rho(x) - N \right)$, where μ is a Lagrange multiplier (the chemical potential).

The first term in (3) is proportional to $\rho(x)$. Therefore, its singular coefficient $-\frac{\lambda}{2m} \partial_x \left. \frac{P}{x-y} \right|_{y=x}$ amounts to a shift of the chemical potential μ by an infinite constant. The last term in (3) is, of course, a field independent constant - an infinite shift of energy. In order for this paper to be self-contained, we have briefly summarized the derivation of the collective-field Hamiltonian (2) in Appendix C.

It is worth mentioning at this point that the Calogero model enjoys a strong-weak-coupling duality symmetry [13, 14]. At the level of the collective Hamiltonian (2), these duality transformations read

$$\tilde{\lambda} = \frac{1}{\lambda}, \quad \tilde{m} = -\frac{m}{\lambda}, \quad \tilde{\mu} = -\frac{\mu}{\lambda}; \quad \tilde{\rho}(x) = -\lambda\rho(x), \quad \text{and} \quad \tilde{\pi}(x) = -\frac{\pi(x)}{\lambda}, \quad (8)$$

and it is straightforward to see that these transformations leave (2) (including the chemical potential term) invariant. The minus signs which occur in (8) are all important: We interpret all negative values of the parameters and densities as those pertaining to holes, or antiparticles. Thus, the duality transformations (8) exchange particles and antiparticles. (For more details see e.g. Section 3 of [4], and references therein.)

It is well-known [9] that to leading order in the $\frac{1}{N}$ expansion, collective dynamics of our system is determined by the classical equations of motion resulting from (2). The simplest solution of these equations is the constant condensate $\rho(x) = \rho_0$ (and $\pi(x) = 0$) corresponding to the ground state.

More interesting solutions of these equations include various types of periodic density waves and soliton configurations [15, 16, 17]. As we explain in Section 2

below, these periodic density waves can be thought of as a crystal made of the localized soliton solution.

Recently, density wave configurations of this type were studied, among other things, in [18], where a certain regulator, first introduced in [16], was used to tame the effective collective potential. The BPS-equations associated with the regulated potential were then converted into a Riccati equation, which was solved explicitly.

Such static periodic density waves are the focus of the present paper as well. As in [18], we convert the BPS-equations associated with the equations of motion of (2) into an explicitly solvable Riccati equation. However, unlike [18], we avoid introducing any unconventional regulators in (2). In addition, we also construct non-BPS solutions of the equations of motion, which are simple shifts of BPS-solutions by a constant, and compute their energy densities. That constant is fixed by the equation of motion and turns out to be either the maximum or the minimum value of the corresponding BPS-solution. Thus, these non-BPS density profiles vanish periodically and coincide with the large-amplitude waves reported in [17], albeit without too many details of their construction. We believe that the constructive way in which we derive our static periodic BPS and non-BPS configurations complements the discussion in [15, 17, 18]. Since these non-BPS solutions vanish periodically, we can also refer to them as vortex crystals, as they constitute a periodic generalization of the vortex solution of [16].

In the present paper we also show how these known solitary and periodic wave-solutions appear in the collective field theory of the two-family generalization of the CM, under very special conditions on the coupling constants. The two-family Calogero model is a generalization of (1) into two species of identical particles. The

Hamiltonian of this model reads [19]

$$H = -\frac{1}{2m_1} \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i{}^2} + \frac{\lambda_1(\lambda_1 - 1)}{2m_1} \sum_{i \neq j}^{N_1} \frac{1}{(x_i - x_j)^2}$$

$$-\frac{1}{2m_2} \sum_{\alpha=1}^{N_2} \frac{\partial^2}{\partial x_\alpha{}^2} + \frac{\lambda_2(\lambda_2 - 1)}{2m_2} \sum_{\alpha \neq \beta}^{N_2} \frac{1}{(x_\alpha - x_\beta)^2}$$

$$+ \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\lambda_{12}(\lambda_{12} - 1) \sum_{i=1}^{N_1} \sum_{\alpha=1}^{N_2} \frac{1}{(x_i - x_\alpha)^2} \,. \tag{9}$$

Here, the first family contains N_1 particles of mass m_1 at positions x_i, $i = 1, 2, ..., N_1$, and the second one contains N_2 particles of mass m_2 at positions x_α, $\alpha = 1, 2, ..., N_2$. All particles interact via two-body inverse-square potentials. The interaction strengths within each family are parametrized by the coupling constants λ_1 and λ_2, respectively. The interaction strength between particles of the first and the second family is parametrized by λ_{12}.

In (9) we imposed the restriction that there be no three-body interactions, which requires [19]-[24]

$$\frac{\lambda_1}{m_1{}^2} = \frac{\lambda_2}{m_2{}^2} = \frac{\lambda_{12}}{m_1 m_2}. \tag{10}$$

It follows from (10) that

$$\lambda_{12}^2 = \lambda_1 \lambda_2 \,. \tag{11}$$

We assume that (10) and (11) hold throughout this paper wherever we discuss the two-family CM. The Hamiltonian (9) describes the simplest multi-species Calogero model for particles on the line, interacting only with two-body potentials.

In [4] we studied the collective field theory of the two-family CM. The corresponding collective Hamiltonian is

$$H_{coll} = \frac{1}{2m_1} \int dx \, \partial_x \pi_1(x) \, \rho_1(x) \, \partial_x \pi_1(x)$$

$$+ \quad \frac{1}{2m_1} \int dx \rho_1(x) \left(\frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} + \lambda_1 \!\!\fint\!\! \frac{dy \rho_1(y)}{x-y} + \lambda_{12}\!\!\fint\!\! \frac{dy \rho_2(y)}{x-y} \right)^2$$

$$+ \quad \frac{1}{2m_2} \int dx \, \partial_x \pi_2(x) \, \rho_2(x) \, \partial_x \pi_2(x)$$

$$+ \quad \frac{1}{2m_2} \int dx \rho_2(x) \left(\frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} + \lambda_2 \!\!\fint\!\! \frac{dy \rho_2(y)}{x-y} + \lambda_{12}\!\!\fint\!\! \frac{dy \rho_1(y)}{x-y} \right)^2$$

$$+ \quad H_{sing}\,, \tag{12}$$

which is a straightforward generalization of (2). Here $\rho_a(x)$ are the collective density fields of the ath family ($a = 1, 2$), and $\pi_a(x)$ are their conjugate momenta. As in (2), the term H_{sing} denotes a singular contribution which is a straightforward generalization of the one-family expression (3). Given that there are N_a particles in the ath family, the densities $\rho_a(x)$ must be normalized according to

$$\int_{-\infty}^{\infty} dx \, \rho_1(x) = N_1\,, \qquad \int_{-\infty}^{\infty} dx \rho_2(x) = N_2\,. \tag{13}$$

As in the one-family case, these normalization conditions are implemented by adding to (12) the chemical-potential terms $\sum_{a=1,2} \mu_a \left(\int_{-\infty}^{\infty} dx \, \rho_a(x) - N_a \right)$.

As was discussed in [4], the collective Hamiltonian (12) is invariant under an Abelian group of strong-weak-coupling dualities, which is a generalization of the single-family case (8). A remarkable consequence of these duality symmetries (see Section 3.1 of [4] for more details) is that when one sets

$$\lambda_1 \lambda_2 = 1\,, \qquad \lambda_{12} = -1 \tag{14}$$

in (10), the two-family CM (12) becomes similar, in some sense, at the level of collective field theory, to the original single family CM, with a collective Hamiltonian

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effectively given by (2), for a single effective density. More precisely, this similarity manifests itself in the fact that at the special point (14), the original Hamiltonian (12) can be mapped by these duality symmetries onto a two-family collective Hamiltonian in which the two families are still *distinct*, but have common mass and two-body interaction couplings, and therefore, the two densities can be combined into a certain effective one-family density ρ_{eff}. In fact, at these special points, the classical densities (i.e., the static solutions $\rho_1(x)$ and $\rho_2(x)$ of the equations of motion associated with (12)) turn out to be proportional to each other, and of opposite signs. Thus, for example, for $m_2 = -\frac{m_1}{\lambda_1} < 0$, the common parameters mentioned above are $\lambda = \lambda_1$ and $m = m_1$, leading to an effective one-family density

$$\rho_{eff} = \rho_1 - \frac{\rho_2}{\lambda_1}, \tag{15}$$

which satisfies the static equation of motion of the single-family model (2) with these common parameters, whereas for $m_1 = -\frac{m_2}{\lambda_2} < 0$, one obtains similar relations, but with the two families interchanged. (Negative masses and densities in these formulas are interpreted as quantities corresponding to holes rather than particles, as was mentioned above.)

In conclusion of this introduction it is proper to mention that the Heisenberg equations of motion of the collective field $\rho(x)$ and its conjugate momentum $\pi(x)$ may be interpreted as the isentropic hydrodynamic flow equations of an Eulerian fluid [25] (see also [10]) and the latter may be associated with the completely integrable and soliton-bearing Benjamin-Ono equation, both at the classical level [26, 27] and the quantum level [28].

This paper is organized as follows: In Section 2 we solve the static BPS equation associated with the one-family collective Hamiltonian (2) by converting it into a

Riccati equation which we then solve explicitly. The solution is a static periodic density wave - the finite amplitude wave solution of [15]. Conversion of the BPS equation into a Riccati equation is achieved by considering the complex valued resolvent $\Phi(z)$ associated with the positions of the N particles on the line (see Eq.(20)), whose boundary value, as the complex variable z approaches the real axis, is a linear combination of the density field $\rho(x)$ and its Hilbert-transform $\rho^H(x)$ (see Eq. (22)) [18, 28]. That the latter combination satisfies the Riccati equation then follows from the BPS-equation and its Hilbert-transform. We then study various limits of the periodic solution. We conclude Section 2 by showing that the coupled BPS-equations, associated with (12) at the special point (14) in parameter space do indeed collapse into a single-family BPS equation.

In Section 3 we consider the static limit of the equation of motion associated with (2) - namely, the full variational equation. Every solution of the BPS-equation is, of course, a solution of the full variational equation. It is more challenging to find non-BPS solutions of the latter. We seek such solutions in the form of BPS configurations shifted by a constant, as was mentioned above. For each of the cases $\lambda > 1$ and $0 < \lambda < 1$ we find two types of solutions, namely, a positive periodic density wave (a vortex crystal) and a negative one (an anti-vortex crystal). We discuss how these solutions map onto each other by the duality transformations (8). Then, we discuss the energy density of these non-BPS solutions, averaged over a period. We end Section 3 by showing that the coupled variational equations, associated with (12) at the special point (14) in parameter space collapse into a single-family variational equation.

For the sake of completeness, and also for future use, we provide and prove in

Appendix A a compendium of useful identities involving Hilbert-transforms.

In Appendix B we note and also resolve a mathematical paradox associated with the variational equation. It has to do with the trilocal term in the density fields obtained by expanding the square in (2). In many papers on the collective approach to the Calogero model, that trilocal term is converted into a local $\rho^3(x)$ term by employing a certain identity among distributions, Eq.(A.20). However, strictly speaking, that identity is valid only for distributions acting on test functions which are integrable along the whole real line. The periodic density profiles discussed in this paper are certainly not of this type. Nevertheless, they arise correctly as solutions of the variational equation associated with the alternative form of the collective potential containing the $\rho^3(x)$ term, given in Eq.(B.2), as they do, for example, in the pioneering work [15], where these periodic density waves were discovered. The resolution of this paradox lies in proper readjustment of the chemical potential enforcing the constraint (7).

Finally, in order for this paper to be self-contained, we briefly summarize in Appendix C the derivation of the collective-field Hamiltonian (2) from (1).

2 Periodic BPS Density Waves: Soliton Crystals

The Hamiltonian (2) is essentially the sum of two positive terms[2]. Its zero-energy classical solutions are zero-momentum, and therefore time independent configura-

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[2]Recall the constraint (7) and our comment concerning H_{sing} following (7). In addition, as was mentioned above, the external confining potential was set to zero. Thus, the first two terms in (2) comprise the BPS limit of the model.

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tions of the collective field (4), which are also solutions of the BPS equation

$$B[\rho] \equiv \frac{\lambda - 1}{2} \frac{\partial_x \rho}{\rho} + \lambda \fint \frac{dy \rho(y)}{x - y} = 0 \,. \tag{16}$$

It is easy to check that the duality transformation (8) maps a solution $\rho(x)$ of (16) with coupling λ onto another solution $\tilde{\rho}(x) = -\lambda \rho(x)$ of that equation with coupling $\tilde{\lambda} = \frac{1}{\lambda}$. As we shall see below in Eq. (33), all solutions of (16) are of definite sign, and never vanish along the real axis. Thus, such a positive solution of (16) is mapped by (8) onto a negative solution, and vice-versa.

The BPS equation (16) may be written alternatively as

$$(\lambda - 1)\, \partial_x \rho = 2\pi \lambda \rho \rho^H \,, \tag{17}$$

where ρ^H is the Hilbert-transform (A.1) of ρ. Note that for $\lambda = 1$, where the CM describes non-interacting fermions[3] the only solution of (17) is $\rho = \rho_0 = \text{const.}$ Henceforth, we shall assume $\lambda \neq 1$.

The proper way to solve this nonlinear integro-differential equation is to consider it together with its Hilbert-transform[18, 28]

$$(\lambda - 1)\, \partial_x \rho^H = \pi \lambda ((\rho^H)^2 - \rho^2 + \rho_0^2) \,, \tag{18}$$

where on the RHS we used the identity (A.14) (and the fact that $\partial_x \rho^H = (\partial_x \rho)^H$ on the LHS). Here ρ_0 is a real parameter such that

$$\int\limits_{-\infty}^{\infty} dx\, (\rho(x) - \rho_0) = 0 \,. \tag{19}$$

It arises from the fact that we seek a solution of $\rho(x)$ which need not necessarily decay at spatial infinity. (See (A.2).) Note that (18) is even in ρ_0. By definition,

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[3]The constant solution is also the sole solution of (17) when $\lambda = 0$, corresponding to non-interacting bosons.

the sign of ρ_0 coincides with that of $\rho(x)$, the solution of (18). A positive solution $\rho(x) \geq 0$ corresponds to a BPS configuration of particles, and a negative one - to a configuration of antiparticles, as was mentioned following (8).

To arrive at the Riccati equation mentioned in the introduction, we proceed as follows. Given the density $\rho(x)$, consider the resolvent

$$\Phi(z) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dy\, \frac{\rho(y)}{z-y} \tag{20}$$

associated with it, in which z is a complex variable. It is easy to see that it is related to the resolvent $G(z)$ of the *subtracted* density $\bar{\rho}(x) = \rho(x) - \rho_0$, defined in (A.3), by

$$\Phi(z) = -\frac{1}{\pi} G(z) + i\rho_0 \,\mathrm{sign}\,(\Im z)\,. \tag{21}$$

The resolvent $\Phi(z)$ is evidently analytic in the complex plane, save for a cut along the support of $\rho(x)$ on the real axis. From the identity (A.4) we obtain

$$\Phi_{\pm}(x) \equiv \Phi(x \pm i0) = \rho^H(x) \pm i\rho(x)\,, \tag{22}$$

consistent with (21) and (A.5). Thus, if $\Phi(z)$ is known, $\rho(x)$ can be determined from the discontinuity of $\Phi(z)$ across the real axis.

An important property of $\Phi(z)$, which follows directly from the definition (20), is

$$\Im \Phi(z) = \frac{\Im z}{\pi} \int_{-\infty}^{\infty} \frac{\rho(y)\, dy}{|z-y|^2}\,. \tag{23}$$

Thus, if $\rho(x)$ does not flip its sign throughout its support, we have

$$\mathrm{sign}\,(\Im \Phi(z)) = \mathrm{sign}\,(\Im z)\,\mathrm{sign}\,(\rho(x))\,. \tag{24}$$

We shall use this property to impose certain further conditions on the solution of (26) below.

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It follows from (22) that (17) and (18) are, respectively, the imaginary and real parts of the Riccati equation

$$(\lambda - 1)\partial_x \Phi_\pm(x) = \pi\lambda(\Phi_\pm^2(x) + \rho_0^2) \tag{25}$$

obeyed by both complex functions $\Phi_\pm(x)$. Let $\Phi_\pm(z)$ be the analytic continuations of $\Phi_\pm(x)$ into the $z-$upper and lower half planes, respectively. These functions are evidently the two solutions of

$$(\lambda - 1)\partial_z \Phi(z) = \pi\lambda(\Phi(z)^2 + \rho_0^2), \tag{26}$$

subjected to the boundary conditions $\Phi_+^*(x+i0) = \Phi_-(x-i0)$ and sign $(\Im\Phi_+(x + i0)) =$ sign $(\rho(x)) =$ sign ρ_0, from (22). The resolvent (20) is then obtained by patching together $\Phi_+(z)$ in the upper half-pane and $\Phi_-(z)$ in the lower half-plane.

The standard way to solve (26) is to write it as

$$\left(\frac{1}{\Phi(z) - i\rho_0} - \frac{1}{\Phi(z) + i\rho_0}\right)\partial_z\Phi(z) = ik, \tag{27}$$

where

$$k = \frac{2\pi\lambda\rho_0}{\lambda - 1}, \tag{28}$$

is a real parameter.

Straightforward integration of (27) then yields the solutions

$$\Phi_\pm(z) = i\rho_0 \frac{1 + e^{ikz - u_\pm}}{1 - e^{ikz - u_\pm}}, \tag{29}$$

where $u_\pm$ are integration constants. The boundary condition $\Phi_+^*(x + i0) = \Phi_-(x - i0)$ then tells us that $u_- = -u_+^*$. Clearly, $\Im u_+$ can be absorbed by a shift in x. Therefore, with no loss of generality we set $\Im u_+ = 0$. The second boundary

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condition sign $(\Im \Phi_+(x + i0)) = \text{sign} \, \rho_0$ then tells us that $u \equiv \Re u_+ > 0$. Thus, $\Phi_\pm(z)$ are completely determined and we obtain (20) as

$$\Phi(z) = i\rho_0 \frac{1 + e^{ikz - u \, \text{sign} \, (\Im z)}}{1 - e^{ikz - u \, \text{sign} \, (\Im z)}} . \tag{30}$$

As can be seen in (33) below, the density $\rho(x)$ associated with (30) is indeed of definite sign, namely, sign ρ_0.

The asymptotic behavior of (30) is such that

$$\Phi(\pm i\infty) = \pm i\rho_0 \, \text{sign} \, k . \tag{31}$$

This must be consistent with (24), which implies (together with the fact that sign $(\rho(x)) = \text{sign} \, \rho_0$) that k must be *positive*. In other words, as can be seen from (28), positive (space-dependent) BPS density configurations ($\rho_0 > 0$) exist only for $\lambda > 1$, and negative (space-dependent) BPS densities ($\rho_0 < 0$) arise only for $0 < \lambda < 1$[4]. The duality symmetry (8), which interchanges the domains $0 < \lambda < 1$ and $\lambda > 1$, maps these two types of BPS configurations onto each other.

Now that we have determined $\Phi(z)$, let us extract from it the BPS density $\rho(x)$ and its Hilbert transform $\rho^H(x)$. From (30) we find that

$$\Phi_+(x) = \Phi(x + i0) = \rho_0 \frac{-\sin kx + i \sinh u}{\cosh u - \cos kx} , \tag{32}$$

from which we immediately read-off the solution of the BPS-equation (16) as

$$\rho(x) \quad = \quad \rho_0 \frac{\sinh u}{\cosh u - \cos kx}$$

$$\rho^H(x) \quad = \quad -\rho_0 \frac{\sin kx}{\cosh u - \cos kx} , \tag{33}$$

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[4]Constant solutions $\rho = \rho_0$ of (16), are of course not subjected to this correlation between sign ρ_0 and the range of λ.

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where both $k > 0$ and $u > 0$, and the sign of $\rho(x)$ coincides with that of ρ_0. That ρ^H in (33) is indeed the Hilbert-transform of ρ can be verified by explicit calculation.

The static BPS density-wave, given by $\rho(x)$ in (33), is nothing but the finite-amplitude solution of [15]. It comprises a two-parameter family of spatially periodic solutions, all of which have zero energy density, by construction. The period is

$$T = \frac{2\pi}{k} = \frac{\lambda - 1}{\lambda \rho_0}. \tag{34}$$

It can be checked by explicit calculation[5] that

$$\frac{1}{T} \int_{\text{period}} \rho(x)\, dx = \rho_0, \tag{35}$$

and therefore that $\int_{-\infty}^{\infty} (\rho(x) - \rho_0)\, dx = 0$, as required by definition of ρ_0. Thus, the parameter ρ_0 determines both the period of the solution $\rho(x)$, as well as its period-average, and the other (positive) parameter u determines the amplitude of oscillations about the average value. Note also from (35), that the number of particles per period is

$$T \rho_0 = \frac{\lambda - 1}{\lambda}. \tag{36}$$

A couple of limiting cases of (33) are worth mentioning. Thus, if we let $u \to 0$, we obtain a comb of Dirac $\delta-$functions

$$\rho(x) = \frac{\lambda - 1}{\lambda} \sum_{n \in \mathbb{Z}} \delta(x - nT). \tag{37}$$

If, in addition to $u \to 0$, we also let k tend to zero (or equivalently, let the period T diverge), such that $b = \frac{u}{k}$ remains finite, we obtain the BPS soliton solution [15, 16]

$$\rho(x) = \frac{\lambda - 1}{\lambda} \frac{1}{\pi} \frac{b}{b^2 + x^2}. \tag{38}$$

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[5]The best way to do this computation is to change variables to $t = e^{ikx}$ and transform the integral into a contour integral around the unit circle.

In fact, the original construction of the periodic soliton (33) in [15] was done by juxtaposing infinite solitons like (38) in a periodic array. For this reason we may refer to the finite amplitude BPS density wave in (33) also as the *soliton crystal*.

Note that the relation (36) is preserved in both limiting cases, since the RHS of (36) depends neither on u nor on k.

2.1 BPS Solutions of the Two-Family Model at the Special Point (14)

The BPS-equations of the two-family collective field Hamiltonian (12) are

$$B_1[\rho_1, \rho_2] \equiv \frac{\lambda_1 - 1}{2} \frac{\partial_x \rho_1}{\rho_1} + \lambda_1 f \frac{dy \rho_1(y)}{x - y} + \lambda_{12} f \frac{dy \rho_2(y)}{x - y} = 0$$

$$B_2[\rho_1, \rho_2] \equiv \frac{\lambda_2 - 1}{2} \frac{\partial_x \rho_2}{\rho_2} + \lambda_2 f \frac{dy \rho_2(y)}{x - y} + \lambda_{12} f \frac{dy \rho_1(y)}{x - y} = 0 \,. \tag{39}$$

Solutions of these coupled equations yield the time-independent zero-energy and zero-momentum configurations of the collective fields ρ_1 and ρ_2.

Finding the general solution of these coupled equations for arbitrary couplings and masses (subjected to (10)) is still an open problem, which we do not address in the present paper. However, at the special point (14), where the two-family model becomes similar to a single-family model, the two equations (39) simplify drastically, becoming linearly dependent. For example, for

$$\lambda = \lambda_1 = \frac{1}{\lambda_2}, \quad \lambda_{12} = -1, \quad \text{and} \quad m = m_1 = -\lambda m_2, \tag{40}$$

it is easy to see that

$$B_1 + \lambda B_2 = \frac{\lambda - 1}{2} \partial_x \log\left(\frac{\rho_1}{\rho_2}\right). \tag{41}$$

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Since at the same time, from (39), $B_1 = B_2 = 0$, (41) implies that the two densities must be proportional

$$\rho_2(x) = -\kappa \rho_1(x). \tag{42}$$

(From the discussion in [4] we know that the constant $\kappa > 0$, and the negative density is interpreted as density of holes, as was mentioned in the Introduction.) Upon substitution of (42) back in (39) we see that

$$B_1 = \frac{\lambda - 1}{2} \frac{\partial_x \rho_{eff}}{\rho_{eff}} - \lambda \pi \rho_{eff}^H, \tag{43}$$

coincides with the corresponding one-family expression B in (16) with an effective density ρ_{eff} given by (15). Thus, at this special point, $\rho_{eff}(x)$ is given by (33), from which ρ_1 and ρ_2, being proportional to ρ_{eff}, can be deduced as well. An analogous solution of (39) exists for the case in which the roles of the two families in (40) are interchanged.

3   Non-BPS Solutions of the Equation of Motion

The uniform-density ground state, as well as the periodic space-dependent BPS-configurations discussed in the previous section, all correspond to zero-energy and zero-momentum configurations of the collective field Hamiltonian (2). Static density configurations with *positive* energy density are found by extremizing the collective potential

$$
\begin{aligned}
V_{coll} &= \frac{1}{2m} \int dx\, \rho(x) \left(\frac{\lambda - 1}{2} \frac{\partial_x \rho}{\rho} + \lambda \!\!\int \frac{dy\, \rho(y)}{x - y} \right)^2 + \mu \left(N - \int dx\, \rho(x) \right) \\
&= \frac{1}{2m} \int dx\, \rho(x) B[\rho]^2 + \mu \left(N - \int dx\, \rho(x) \right)
\end{aligned}
\tag{44}
$$

part of (2).

Computation of the variation of (44) with respect to ρ is most easily carried with the help of (45) just below. Thus, using the elementary relation $\int\, dx\, F(x)\, G^H(x) = -\int\, dx\, F^H(x)\, G(x)$ it is easy to obtain the variational identity

$$\int\, dx\, \rho(x)\, F(x)\, \delta B[\rho] = \int\, dx\, \left[-\frac{\lambda - 1}{2\rho}\, \partial_x(\rho F) + \pi\lambda\, (\rho F)^H \right] \delta\rho(x)\,, \qquad (45)$$

where the infinitesimal variation of $B[\rho]$

$$\delta B[\rho] = \frac{\lambda - 1}{2}\, \partial_x \left(\frac{\delta\rho}{\rho} \right) - \pi\lambda\, \delta\rho^H \qquad (46)$$

was computed from (16).

Using (45), it is straightforward to obtain the desired variational equation as

$$2m\, \frac{\delta V_{coll}}{\delta\rho(x)} = B[\rho]^2 - \frac{\lambda - 1}{\rho}\, \partial_x(\rho B[\rho]) + 2\pi\lambda\, (\rho B[\rho])^H - 2m\mu = 0\,. \qquad (47)$$

The collective potential (44) is invariant under the duality transformation (8). Thus, (47) must transform covariantly under (8). Indeed, it is straightforward to see that under (8), the variational equation (47) transforms into $\frac{1}{\lambda^2}$ times itself. In this way, a solution $\rho(x)$ of (47) with parameters λ, m, μ is transformed into a solution $\tilde{\rho}(x)$ of (47) with parameters $\tilde{\lambda}, \tilde{m}, \tilde{\mu}$. We shall make use of this fact later-on.

Evidently, any solution of the BPS equation $B[\rho] = 0$ (Eq.(16)) is also a solution of the variational equation (47) (with $\mu = 0$), reflecting the fact that (44) is quadratic and homogeneous in $B[\rho]$.

Unfortunately, we do not know how to find the most general solution of this equation. Therefore, we shall content ourselves with finding a particular family of solutions to (47) of the simple shifted form

$$\rho(x) = \rho_s(x) + c\,, \qquad (48)$$

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where

$$\rho_s(x) = \rho_0 \frac{\sinh u}{\cosh u - \cos kx} \tag{49}$$

is the BPS profile in (33) and c is an unknown constant, to be determined from (47). (Clearly, (48) with $c = 0$ must be a solution of (47).)

Let us proceed in a few steps. First, note that

$$\rho^H(x) = \rho_s^H(x) = -\rho_0 \frac{\sin kx}{\cosh u - \cos kx}, \tag{50}$$

from (33). Then, compute

$$B[\rho] = \frac{\lambda - 1}{2} \frac{\partial_x \rho}{\rho} - \lambda \pi \rho^H = \frac{\lambda \pi c \rho_0 \sin \, kx}{c(\cosh \, u - \cos \, kx) + \rho_0 \sinh u}, \tag{51}$$

from which we obtain the remarkably simple relation

$$\rho \, B[\rho] = -\lambda \pi c \rho^H. \tag{52}$$

Therefore,

$$(\rho \, B[\rho])^H = \lambda \pi c (\rho_s - \rho_0), \tag{53}$$

where we used the identity (A.8). (Note that (53) is consistent with the identity $\int_{-\infty}^{\infty} F^H(x) \, dx = 0$.) Substituting the ansatz (48) and the auxiliary results (51)-(53) in (47) we obtain the LHS of the latter as a rational function of polynomials in $\cos kx$. The numerator of that function is a cubic polynomial, which we then expand into a finite cosine Fourier series, all coefficients of which must vanish. Thus, the coefficient of $\cos 3kx$ determines the chemical potential in terms of the remaining parameters as

$$\mu = -\frac{(\lambda \pi)^2}{2m} \rho_0 (\rho_0 + 2c), \tag{54}$$

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which we then feed into the coefficients of the remaining three terms. The coefficient of $\cos 2kx$ is then found as the cubic

$$-(\lambda\pi)^2 \rho_0\, c(c^2 \sinh\, u + 2c\rho_0 \cosh\, u + \rho_0^2 \sinh\, u)\,, \tag{55}$$

where we used (28) on the way. This coefficient must vanish, yielding a cubic equation for c. The remaining Fourier coefficients vanish identically upon substitution of the roots of this cubic equation for c.

As we have anticipated following (48), one root of this cubic equation is obviously $c_0 = 0$, which corresponds to $\rho = \rho_s$. The other two roots are

$$c_1 = -\rho_0\, \tanh\, \frac{u}{2} \qquad \text{and} \qquad c_2 = -\rho_0\, \coth\, \frac{u}{2}\,. \tag{56}$$

Note that neither of these roots, and therefore neither of the shifted solutions (48), depend on m or on μ. Once the parameters m and μ are related according to (54), they drop out of any further consideration.

3.1 Large Amplitude Density Waves: Vortex Crystal Solutions

From this point onward we shall discuss the cases $\lambda > 1$ and $0 < \lambda < 1$ separately.

3.1.1 The case $\lambda > 1$

In this case $\rho_0 > 0$, as we saw following (31). For positive ρ_0, the first root c_1 in (56) amounts in (48) to shifting the BPS solution $\rho_s(x)$ by its *minimum*. The resulting solution

$$\rho_p(x) = \rho_0 \left(\frac{\sinh u}{\cosh u - \cos kx} - \tanh\, \frac{u}{2} \right) \tag{57}$$

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is a positive function which vanishes periodically. We shall refer to it as the *vortex crystal* solution, as it is a periodic generalization of the single vortex solution of [16]. Since $\rho_p(x) > 0$, it is a density of particles (rather than holes). Therefore it corresponds to having a positive mass parameter $m > 0$ in (44). The vortex crystal (57) coincides with the so-called *large amplitude* wave solution of [17] for the case $\lambda > 1$ and zero velocity.

The second root c_2 in (56) amounts to shifting the BPS solution $\rho_s(x)$ by its *maximum*. The resulting solution

$$\rho_n(x) = \rho_0 \left(\frac{\sinh u}{\cosh u - \cos kx} - \coth \frac{u}{2} \right) \tag{58}$$

is thus a negative function which vanishes periodically - an anti-vortex crystal. Being a negative solution of (47), Eq.(58) should be interpreted as the density of holes rather than particles. Therefore it corresponds to having a negative mass $m < 0$ in (44).

3.1.2 The case $0 < \lambda < 1$

In this case $\rho_0 < 0$, as we saw following (31). Therefore c_1 and c_2 in (56) switch roles: For negative ρ_0, c_1 amounts to shifting in (48) the BPS solution $\rho_s(x)$ by its *maximum*. The resulting solution

$$\tilde{\rho}_n(x) = \rho_0 \left(\frac{\sinh u}{\cosh u - \cos kx} - \tanh \frac{u}{2} \right) \tag{59}$$

is a negative function which vanishes periodically - an anti-vortex crystal. It is therefore a density of holes corresponding to having a negative mass $m < 0$ in (44).

The second root c_2 amounts in this case to shifting the BPS solution $\rho_s(x)$ by

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its *minimum*. The resulting solution

$$\tilde{\rho}_p(x) = \rho_0 \left(\frac{\sinh u}{\cosh u - \cos kx} - \coth \frac{u}{2} \right) = |\rho_0| \left(\coth \frac{u}{2} - \frac{\sinh u}{\cosh u - \cos kx} \right) \quad (60)$$

is thus a positive function which vanishes periodically - a vortex crystal. It corresponds to having $m > 0$ in (44), in a similar manner to $\rho_p(x)$ in (57). $\tilde{\rho}_p(x)$ coincides with the large amplitude wave solution of [17] for the case $0 < \lambda < 1$ and zero velocity. Note that $\tilde{\rho}_p(x)$ has appeared also in [18].

The duality transformations (8) leave the wave-number k in (28) invariant. By definition, the positive parameter u, defined in (30), is invariant under (8) as well. Thus, evidently, the duality transformations (8) map $\rho_p(x)$ in (57) and $\tilde{\rho}_n(x)$ in (59) onto each other. (Of course, the ρ_0 parameters appearing in the latter two equations are different from each other, and related by the fourth relation in (8).) Similarly, the duality transformations (8) map $\rho_n(x)$ in (58) and $\tilde{\rho}_p(x)$ in (60) onto each other.

3.1.3 Average Energy Densities per Period

Our new solutions (57) - (60) of the variational equation (47) are periodic functions, with the same period as that of the BPS solution ρ_s. Since these are non-BPS configuration, they must carry positive energy density[6]. We shall now proceed to calculate the mean energy densities per period of these configurations, to which end we must determine the combination $\rho B[\rho]^2$ appearing in (44). From the general expressions (51) and (52) we obtain

$$\rho B[\rho]^2 = \frac{\lambda \pi c \rho_0 \sin kx}{\cosh u - \cos kx} \frac{\lambda \pi c \rho_0 \sin kx}{c(\cosh u - \cos kx) + \rho_0 \sinh u}. \quad (61)$$

---

[6]Since negative densities correspond to holes, whose mass should be taken negative, we have $\frac{\rho(x)}{m} > 0$ in these cases as well. This renders V_{coll} in (44) positive for such densities. Thus, the negative solutions ρ_n and $\tilde{\rho}_n$ carry positive energy density, as their positive counterparts obviously do.

The desired period-averaged energy density is then given by

$$\mathcal{E} = \frac{1}{T} \int\limits_{\text{period}} dx \, \frac{\rho B[\rho]^2}{2m} \,, \tag{62}$$

where $T = \frac{2\pi}{k}$ (Eq. (34)).

We shall content ourselves with computing the energy density only of the positive densities ρ_p and $\tilde{\rho}_p$. In order to compute $\mathcal{E}$ of (57), corresponding to $\lambda > 1$ and $\rho_0 > 0$, we substitute $c = c_1$ in (61). After some algebra, we find that in this case

$$\rho_p B[\rho_p]^2 = (\lambda \pi \rho_0)^2 \tanh \frac{u}{2} \left[\rho_0 (1 - \tanh \frac{u}{2}) - (\rho_s(x) - \rho_0) \tanh \frac{u}{2} \right] . \tag{63}$$

In view of (35), and by definition of ρ_0, the period-average of $\rho_s(x) - \rho_0$ vanishes. Thus, from (44), we obtain the period-average energy density of (57) as

$$\mathcal{E}[\rho_p] = \frac{(\lambda \pi \rho_0)^2}{2m} \rho_0 \tanh \frac{u}{2} (1 - \tanh \frac{u}{2}) \,, \tag{64}$$

which is a manifestly positive quantity. It depends continuously on the two parameters ρ_0 and u, comprising an unbounded continuum of positive energies, which is not gapped from the zero energy density of the BPS solitons.

Similarly, in order to compute $\mathcal{E}$ of (60), corresponding to $0 < \lambda < 1$ and $\rho_0 < 0$, we substitute $c = c_2$ in (61). After some algebra, we find that in this case

$$\tilde{\rho}_p B[\tilde{\rho}_p]^2 = -(\lambda \pi \rho_0)^2 \coth \frac{u}{2} \left[\rho_0 (\coth \frac{u}{2} - 1) + (\rho_s(x) - \rho_0) \coth \frac{u}{2} \right] , \tag{65}$$

which leads to the positive period-average energy density of (60) given by

$$\mathcal{E}[\tilde{\rho}_p] = - \frac{(\lambda \pi \rho_0)^2}{2m} \rho_0 \coth \frac{u}{2} (\coth \frac{u}{2} - 1) \,. \tag{66}$$

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3.1.4 Energy Densities at Fixed Average Particle Density

It is particularly useful to consider the energy densities (64) and (66) at a fixed average particle density per period. The latter is, of course, the subtraction constant as defined in (19), which is given by

$$\tilde{\rho}_0 = \rho_0 + c \tag{67}$$

for the shifted solutions (48). Both (64) and (66) depend on the two parameters ρ_0 and u. Holding $\tilde{\rho}_0$ fixed can thus be used to eliminate one of these parameters, which we shall take to be u.

Let us concentrate first on ρ_p in (57), for which $c = c_1 = -\rho_0 \tanh \frac{u}{2}$ (and of course, $\lambda > 1$). Thus,

$$\tilde{\rho}_0 = \rho_0 \left(1 - \tanh \frac{u}{2}\right), \tag{68}$$

which is positive, since $\rho_0 > 0$ in (57). Moreover, $\rho_0 \geq \tilde{\rho}_0$ in this case, since $u > 0$. In terms of this fixed $\tilde{\rho}_0$, we obtain $\mathcal{E}[\rho_p]$ in (64) as

$$\mathcal{E}[\rho_p] = \frac{(\lambda \pi)^2}{2m} \tilde{\rho}_0 \rho_0 (\rho_0 - \tilde{\rho}_0), \quad \rho_0 \geq \tilde{\rho}_0 = \text{fixed}. \tag{69}$$

This energy density vanishes at the minimal possible value of $\rho_0 = \tilde{\rho}_0$, corresponding to $u = 0$, and therefore to the BPS density configuration (37). As ρ_0 increases from its minimal value, the period-average energy density $\mathcal{E}[\rho_p]$ increases monotonically from zero to infinity. Increasing ρ_0 really means increasing the wave number $k = \frac{2\pi\lambda\rho_0}{\lambda - 1}$, i.e., making the density modulation wave-length shorter.

It is interesting to note that in terms of k and $\tilde{\rho}_0$ we can write

$$\mathcal{E}[\rho_p] = \frac{(1 - \lambda)\tilde{\rho}_0}{4} \left(\frac{1 - \lambda}{2m} k^2 + \frac{\lambda \pi \tilde{\rho}_0}{m} k\right), \quad k = \frac{2\pi\lambda}{\lambda - 1}\rho_0 \geq \frac{2\pi\lambda}{\lambda - 1}\tilde{\rho}_0 = \text{fixed}, \tag{70}$$

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where the expression within the brackets is nothing but the dispersion relation for fluctuations around the constant background $\tilde{\rho}_0$ [12].

We can analyze the periodic vortices $\tilde{\rho}_p$ in (60) in a similar manner. For these solutions $c = c_2 = -\rho_0 \coth \frac{u}{2}$ (and of course $0 < \lambda < 1$). Thus,

$$\tilde{\rho}_0 = \rho_0 \left(1 - \coth \frac{u}{2}\right) = |\rho_0| \left(\coth \frac{u}{2} - 1\right), \tag{71}$$

which is again positive, since the allowed range of ρ_0 in (60) is $\rho_0 \leq 0$. For a given value of $\tilde{\rho}_0$, $\rho_0 = -\frac{1}{2}(e^u - 1)\tilde{\rho}_0$ ranges throughout the negative real axis as u ranges throughout the positive one. In terms of this fixed $\tilde{\rho}_0$, we obtain an expression for $\mathcal{E}[\tilde{\rho}_p]$ in (66) which coincides with the RHS of (69), but where now $\rho_0 \leq 0$, of course: $\mathcal{E}[\tilde{\rho}_p] = \frac{(\lambda\pi)^2}{2m} \tilde{\rho}_0 |\rho_0| (|\rho_0| + \tilde{\rho}_0)$. This energy density vanishes at the maximal possible value of $\rho_0 = 0$, corresponding to $u = 0$, and therefore to the BPS density configuration (38). As ρ_0 becomes increasingly negative the period-average energy density $\mathcal{E}[\tilde{\rho}_p]$ increases monotonically from zero to infinity. In terms of the wave number k and $\tilde{\rho}_0$, we obtain that $\mathcal{E}[\tilde{\rho}_p] = \frac{(1-\lambda)\tilde{\rho}_0}{4} \left(\frac{1-\lambda}{2m} k^2 + \frac{\lambda\pi\tilde{\rho}_0}{m} k\right)$, which coincides with (70), but where now $k \geq 0$ for any value of $\tilde{\rho}_0 > 0$.

3.2   The Two-Family model at the Special Point (14)

The variational equations associated with the two-family collective potential part of the two-family collective Hamiltonian (12) are

$$B_1^2 - \frac{\lambda_1 - 1}{\rho_1} \partial_x (\rho_1 B_1) + 2\lambda_1 \pi (\rho_1 B_1)^H + 2\frac{m_1}{m_2} \lambda_{12} \pi (\rho_2 B_2)^H - 2m_1 \mu_1 = 0$$

$$B_2^2 - \frac{\lambda_2 - 1}{\rho_2} \partial_x (\rho_2 B_2) + 2\lambda_2 \pi (\rho_2 B_2)^H + 2\frac{m_2}{m_1} \lambda_{12} \pi (\rho_1 B_1)^H - 2m_2 \mu_2 = 0,$$

$$\tag{72}$$

in straightforward analogy with (47), where the BPS combinations B_1 and B_2 were defined in (39).

As in the case of the BPS equations (39), the general solution of these coupled equations for arbitrary couplings and masses (subjected to (10)) is still an open problem, which we do not address in the present paper. However, at the special point (14), where the two-family model becomes similar to a single-family model, the two equations (72) simplify drastically, becoming linearly dependent, in much the same way that the BPS equations (39) got simplified.

Consider, for example, the case (40). In this case (41) still holds, of course, but now neither B_1 nor B_2 need vanish. Thus, we cannot conclude that ρ_1 and ρ_2 must be proportional. Instead, we shall now show that under the condition (40), there is a non-BPS solution of the coupled equations (72) in which the two densities are proportional to each other, as in (42). In this case it follows from (41) that $B_1 + \lambda B_2 = 0$. Substituting this relation as well as (40) in (72) , we see that the two equations coincide, provided $\mu_1 + \lambda \mu_2 = 0$, and that their common form is nothing but the variational equation (47) of the single-family, for an effective density (15). Thus, at this special point, $\rho_{eff}(x)$ is given by (49), (57) or (58), from which ρ_1 and ρ_2, being proportional to ρ_{eff} can be deduced as well. An analogous solution of (72) exists for the case in which the roles of the two families in (40) are interchanged.

Appendix A: A Compendium of Useful Hilbert-Transform Identities

For the sake of completeness, and also for future reference, in this Appendix we list and prove some well-known and useful identities involving Hilbert transforms.

Consider the class of (possibly complex) functions $\rho(x)$ on the whole real line $-\infty < x < \infty$, whose Hilbert transforms

$$\rho^H(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} dy\, \frac{\rho(y)}{y - x} \tag{A.1}$$

exist, and which can be made integrable by subtracting a constant ρ_0. Let us denote

$$\bar{\rho}(x) = \rho(x) - \rho_0 \,. \tag{A.2}$$

(If $\rho(x)$ is already integrable, then $\rho_0 = 0$, of course.)   Thus, for example, if $\rho(x)$ is periodic with period T, with a Fourier zero-mode ρ_0, then $\int_{-\infty}^{\infty} dx\, \bar{\rho}(x) = \int_{-\infty}^{\infty} dx\, (\rho(x) - \rho_0) = 0$.

Given $\bar{\rho}(x)$, consider the resolvent

$$G(z) = \int_{-\infty}^{\infty} dy\, \frac{\bar{\rho}(y)}{z - y} \tag{A.3}$$

associated with it, in which z is a complex variable. The resolvent $G(z)$ is evidently analytic in the complex plane, save for a cut along the support of $\bar{\rho}(x)$ on the real axis. From the identity

$$\frac{1}{x \mp i0} = \frac{P}{x} \pm i\pi\delta(x) \,, \tag{A.4}$$

where P denotes the Cauchy principal value, we then obtain the well-known formula

$$G(x \mp i0) = -\pi \rho^H(x) \pm i\pi \bar{\rho}(x) \,, \tag{A.5}$$

where in the term one before last we used the fact that $\bar{\rho}^H(x) = \rho^H(x)$. Thus, if $G(z)$ is known, $\bar{\rho}(x)$ can be determined from the discontinuity of $G(z)$ across the real axis.

As a nontrivial example consider $\rho(x) = \bar{\rho}(x) = e^{ix}$. For this function $\rho^H(x) = ie^{ix} = i\rho(x)$ and $G(z) = -2\pi i\,\theta(\Im z)\,e^{iz}$. Consequently $G(x-i0) = 0$ and $G(x+i0) = -2\pi i\,e^{ix}$, in accordance with (A.5). As yet another example consider the Cauchy probability distribution $\rho(x) = \bar{\rho}(x) = \frac{\gamma}{\pi}\frac{1}{x^2+\gamma^2}$. For this function $\rho^H(x) = -\frac{1}{\pi}\frac{x}{x^2+\gamma^2}$ and $G(z) = \frac{1}{z+i\gamma\operatorname{sign}(\Im z)}$. Consequently $G(x\mp i0) = \frac{x\pm i\gamma}{x^2+\gamma^2}$, in accordance with (A.5).

For all functions in this class, as $z \to \infty$, $G(z)$ tends asymptotically to zero not slower than $\frac{1}{z}$, that is

$$G(z) \underset{z\to\infty}{\sim} \mathcal{O}\left(\frac{1}{z}\right). \tag{A.6}$$

If, in addition, all moments $M_n = \int\limits_{-\infty}^{\infty} dx\, x^n\,\bar{\rho}(x)$, $(n \geq 0)$ of $\bar{\rho}(x)$ exist, then $G(z)$ is the moment generating function of $\bar{\rho}(x)$, namely, it has the large-z expansion

$$G(z) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}.$$

The analyticity properties of $G(z)$ and the bounds on its asymptotic behavior at infinity are at the heart of our derivation of the Hilbert-transform identities to follow.

From this point on, we shall take all functions $\rho(x)$ to be real. For real $\rho(x)$, we deduce from (A.5) that

$$\rho^H(x) = -\frac{1}{\pi}\Re G(x\mp i0) \quad\text{and}\quad \bar{\rho}(x) = \pm\frac{1}{\pi}\Im G(x\mp i0). \tag{A.7}$$

As a warm-up exercise, let us prove the well-known fact that

$$(\rho^H(x))^H = -\bar{\rho}(x) = \rho_0 - \rho(x). \tag{A.8}$$

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Thus, consider

$$
\begin{aligned}
(\rho^H(x))^H &= (\bar{\rho}^H(x))^H = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \frac{P}{y-x} \, \bar{\rho}^H(y) \, dy \\[2mm]
&= \frac{1}{\pi} \int\limits_{-\infty}^{\infty} dy \, \Re \left[\left(\frac{P}{y-x} - i\pi\delta(y-x) \right) \left(\bar{\rho}^H(y) + i\bar{\rho}(y) \right) \right] dy - \bar{\rho}(x) \\[2mm]
&= -\frac{1}{\pi^2} \Re \int\limits_{-\infty}^{\infty} \frac{G(y+i0)}{y-x+i0} \, dy - \bar{\rho}(x) \,,
\end{aligned}
$$

where in the last step we used (A.4) and (A.7). Let us now prove that the last integral vanishes, from which (A.8) follows. To this end, complete the contour of integration in the last integral (namely, the line running parallel to the real axis just above it) by the infinite semi-circle in the upper half-plane $\Im z > 0$, traversed in the positive sense. Let us denote the closed contour thus formed by γ. Due to the asymptotic behavior (A.6) of $G(z)$ we can establish the first equality in

$$
\int\limits_{-\infty}^{\infty} \frac{G(y+i0)}{y-x+i0} \, dy = \oint_{\gamma} dz \, \frac{G(z)}{z-x} = 0 \,,
$$

whereas the second equality follows since the contour γ encompasses no singularity.

We shall now prove the important identity

$$
(\rho_1 \rho_2^H + \rho_1^H \rho_2)^H = \rho_1^H \rho_2^H - \rho_1 \rho_2 + \rho_{10} \rho_{20} \tag{A.9}
$$

obeyed by any two functions $\rho_1(x)$ and $\rho_2(x)$ in the class of functions considered. Our first step in proving (A.9) is to observe that it may be written equivalently as

$$
(\bar{\rho}_1 \bar{\rho}_2^H + \bar{\rho}_1^H \bar{\rho}_2)^H = \bar{\rho}_1^H \bar{\rho}_2^H - \bar{\rho}_1 \bar{\rho}_2 \,. \tag{A.10}
$$

Consider now the contour integral

$$
I = \oint_{\mathcal{C}_{\infty}} \frac{G_1(z) G_2(z)}{z-x} \, \frac{dz}{2\pi i} \,, \tag{A.11}
$$

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where $G_k(z)$ is the resolvent corresponding to $\bar{\rho}_k(x)$ $(k = 1, 2)$, $x \in I\!\!R$, and where $\mathcal{C}_\infty$ is the circle of infinite radius, centered at the origin. Due to the asymptotic behavior (A.6) of the two resolvents, evidently

$$I = 0. \tag{A.12}$$

Since $G_{1,2}(z)$ are analytic off the real axis, we can deform $\mathcal{C}_\infty$ into the positively oriented boundary Γ of an infinitesimal strip around the real axis (namely, the union of a line parallel to the real axis just below it, traversed from $-\infty$ to ∞, with a line parallel to the real axis just above it and traversed in the opposite direction). The contour integral around Γ essentially picks up the imaginary part of the integrand evaluated just above the real axis. Thus, we have

$$0 = I = \oint_\Gamma \frac{G_1(z)G_2(z)}{z - x} \frac{dz}{2\pi i} = -\frac{1}{\pi} \Im \int_{-\infty}^{\infty} \frac{G_1(y + i0)G_2(y + i0)}{y - x + i0} \, dy. \tag{A.13}$$

The last integrand may be written as

$$\pi^2 \left(\frac{P}{y - x} - i\pi \delta(y - x) \right) \prod_{k=1,2} \left(\bar{\rho}_k^H(y) + i\bar{\rho}_k(y) \right),$$

by virtue of (A.4) and (A.7). Upon substituting the last expression in (A.13) and taking the imaginary part, we obtain the desired result (A.10). Note that for $\rho_1 = \rho_2 = \rho$, (A.9) simplifies into

$$2(\rho\rho^H)^H = (\rho^H)^2 - \rho^2 + \rho_0^2. \tag{A.14}$$

Finally, we shall prove an identity involving three functions $\rho_k(x)$ $(k = 1, 2, 3)$ and their Hilbert transforms. Our proof follows essentially the one given in [29, 30] for the case $\rho_1 = \rho_2 = \rho_3$, which is reproduced also in the text-book [8]. Let $G_k(z)$ be the resolvent corresponding to $\bar{\rho}_k(x)$. Consider now the contour integral

$$J = \oint_{\mathcal{C}_\infty} \frac{dz}{2\pi i} G_1(z)G_2(z)G_3(z). \tag{A.15}$$

As in the previous proof, due to the asymptotic behavior (A.6) of the resolvents, evidently

$$J = 0 \, . \tag{A.16}$$

Since the resolvents are analytic off the real axis, we can deform $\mathcal{C}_\infty$ into the contour Γ, as in the previous proof, which picks up the imaginary part of the integrand evaluated just above the real axis. Thus, we have

$$0 = J = -\frac{1}{\pi} \Im \int_{-\infty}^{\infty} G_1(y + i0) G_2(y + i0) G_3(y + i0) \, dy \, . \tag{A.17}$$

The last integrand may be written as

$$-\pi^3 \prod_{k=1}^{3} \left(\bar{\rho}_k^H(y) + i \bar{\rho}_k(y) \right) \, ,$$

by virtue of (A.7). Upon substituting the last expression in (A.17) and taking the imaginary part, we obtain the desired result

$$\int_{-\infty}^{\infty} \left(\bar{\rho}_1^H \bar{\rho}_2^H \bar{\rho}_3 + \bar{\rho}_1^H \bar{\rho}_2 \bar{\rho}_3^H + \bar{\rho}_1 \bar{\rho}_2^H \bar{\rho}_3^H \right) dx = \int_{-\infty}^{\infty} \bar{\rho}_1 \bar{\rho}_2 \bar{\rho}_3 \, dx \, . \tag{A.18}$$

Note that for $\bar{\rho}_1 = \bar{\rho}_2 = \bar{\rho}_3 = \bar{\rho}$, (A.18) simplifies into

$$3 \int_{-\infty}^{\infty} \bar{\rho} (\bar{\rho}^H)^2 \, dx = \int_{-\infty}^{\infty} (\bar{\rho})^3 \, dx \, , \tag{A.19}$$

which is the identity proved in [8, 29, 30].

Since (A.18) holds for any triplet of functions $\bar{\rho}_k$ in the class of functions thus considered, we can write it formally as an identity among distributions acting upon these test functions, namely, the well-known [31] identity

$$\frac{P}{x-y} \frac{P}{x-z} + \frac{P}{y-x} \frac{P}{y-z} + \frac{P}{z-x} \frac{P}{z-y} = \pi^2 \delta(x-y) \delta(x-z) \, . \tag{A.20}$$

In [31], the identity (A.20) was proved using Fourier transforms. For alternative proofs of the identities discussed in this Appendix, and for more information about Hilbert-transform techniques, see Appendix A of [27].

As should be clear from our proof, (A.20) holds only when the distributions on both its sides act upon functions which are integrable on the whole real line. However, this identity is frequently used in the literature on collective field theory beyond its formal domain of validity. For further discussion of this problem see Appendix B, where we show that this transgression is benign, and can be compensated for by readjusting the chemical potential which governs the normalization condition (7).

Appendix B: A Paradox and its Resolution

The expression for the collective potential in (44) contains bilocal as well as trilocal terms in the density. It is customary in the literature to avoid the trilocal terms by applying a standard procedure as follows: The principal value distribution, acting on functions integrable along the whole real line, satisfies the identity (A.20), which we rewrite here for convenience

$$\frac{P}{x-y}\frac{P}{x-z} + \frac{P}{y-z}\frac{P}{y-x} + \frac{P}{z-x}\frac{P}{z-y} = \pi^2\delta(x-y)\delta(x-z)\,. \tag{B.1}$$

Making use of (B.1) in (44), we obtain

$$\tilde{V}_{coll} = \frac{(\lambda\pi)^2}{6m}\int dx\,\rho^3 + \frac{(\lambda-1)^2}{8m}\int dx\,\frac{(\partial_x\rho)^2}{\rho} + \frac{\lambda(\lambda-1)}{2m}\int dx\,\partial_x\rho \fint dy\frac{\rho(y)}{x-y}$$

$$+ \quad \tilde{\mu}\left(N - \int dx\,\rho(x)\right) \tag{B.2}$$

This expression for $\tilde{V}_{coll}$ is evidently devoid of any trilocal terms. (Note that the chemical potential $\tilde{\mu}$ in (B.2) need not coincide with the one in (44), as our notations imply.)

The classical equation of motion which results from varying (B.2) is

$$\frac{(\lambda\pi)^2}{2m}\rho^2 - \frac{(\lambda-1)^2}{8m}\left(\frac{\partial_x\rho}{\rho}\right)^2 - \frac{(\lambda-1)^2}{4m}\partial_x\left(\frac{\partial_x\rho}{\rho}\right) - \frac{\lambda(\lambda-1)}{m}\fint dy\frac{\partial_y\rho(y)}{x-y} = \tilde{\mu}\,. \tag{B.3}$$

It was this form of the equation of motion (rather than (47)), from which the solitons and density waves were derived in the pioneering work [15].

It can be checked that ρ_s in (49), ρ_p in (57) and ρ_n in (58), the solutions of the variational equation (47) of the first form (44) of the collective potential, are also

solutions of (B.3) (albeit, with values of $\tilde{\mu}$ different from those of (54)). That this is true may look surprising, and even paradoxical to some readers, since neither of these solutions is integrable along the whole real line, which is a necessary condition for (B.1) to hold. This should be clear from the proof of (A.20) in Appendix A, but it can also be demonstrated by a simple counter example - just apply both sides of (B.1) on three constant functions and integrate over all coordinates. The LHS would be null, while the RHS would diverge.

In fact, the latter counter example is precisely relevant to determining the ground state of the collective Hamiltonian (2). The uniform ground state density $\rho = \varrho_0$ is a solution of the BPS equation (16), and of course, also of the variational equation (47) with $\mu = 0$. The energy density tied in it is of course null. It is also a solution of the alternative variational equation (B.3) with $\tilde{\mu} = \frac{(\lambda\pi\varrho_0)^2}{2m}$ and energy density (with respect to (B.2)) $\frac{(\lambda\pi)^2\varrho_0^3}{6m}$.

Thus, it seems that using (B.1) beyond its formal domain of validity is a mild transgression, which is compensated for by appropriately readjusting the chemical potential. This is indeed true, as we shall now prove, thus resolving the paradox why (47) and (B.3) always lead to the same solutions. To this end we shall consider all ρ configurations which are simultaneous solutions of (47) and (B.3). Such functions are evidently extrema of $\Delta V = V_{coll} - \tilde{V}_{coll}$. From (44) and (B.2) we obtain

$$\Delta V = \frac{(\lambda\pi)^2}{6m}\Im\int dx\,(\rho^H + i\rho)^3 - (\mu - \tilde{\mu})\int dx\,\rho$$

$$= \frac{(\lambda\pi)^2}{6m}\Im\int dx\,\Phi^3(x + i0) - (\mu - \tilde{\mu})\Im\int dx\,\Phi(x + i0). \qquad \text{(B.4)}$$

(Note that we have omitted from this expression the constant term $(\mu - \tilde{\mu})N$.) Due to the analytic structure of $\Phi(z)$, and as explained in Appendix A, the latter integral

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can be written as the contour integral

$$\Delta V = -\frac{\lambda^2 \pi^3}{6m} \oint_{\mathcal{C}_\infty} \frac{dz}{2\pi i}\, \Phi^3(z) + \pi(\mu - \tilde\mu) \oint_{\mathcal{C}_\infty} \frac{dz}{2\pi i}\, \Phi(z)\,, \tag{B.5}$$

where $\mathcal{C}_\infty$ is a circle of infinite radius, centered at the origin. Note that $\Phi(z)$ need not decay as $z \to \infty$, since $\int dx\, \rho$ may diverge. Thus, in general $\Delta V \neq 0$.

We shall now determine solutions of

$$\frac{\delta \Delta V}{\delta \rho(x)} = 0\,. \tag{B.6}$$

To this end, let us compute

$$\frac{\delta \Phi(z)}{\delta \rho(x)} = -\frac{1}{\pi} \frac{\delta}{\delta \rho(x)} \int_{-\infty}^{\infty} \frac{\rho(u)\, du}{z - u} = -\frac{1}{\pi} \frac{1}{z - x}\,. \tag{B.7}$$

From this we infer that

$$\frac{\delta \Delta V}{\delta \rho(x)} = \frac{(\lambda \pi)^2}{2m} \oint_{\mathcal{C}_\infty} \frac{dz}{2\pi i} \frac{\Phi^2(z)}{z - x} - (\mu - \tilde\mu)\,. \tag{B.8}$$

The contour $\mathcal{C}_\infty$ in the last integral can be deformed to the countour Γ, defined in Appendix A, which essentially picks up the imaginary part of the integrand evaluated just above the real axis. Thus, in a manner similar to the discussion in Appendix A, from (A.9) to (A.14), we obtain

$$\frac{\delta \Delta V}{\delta \rho(x)} = \frac{(\lambda \pi)^2}{2m} \left[(\rho^H)^2 - \rho^2 - (2\rho \rho^H)^H \right] - (\mu - \tilde\mu)\,. \tag{B.9}$$

But from the identity (A.14) we see that the latter equation boils down to

$$\frac{\delta \Delta V}{\delta \rho(x)} = \tilde\mu - \mu - \frac{(\lambda \pi \rho_0)^2}{2m}\,. \tag{B.10}$$

In other words, the condition (B.6) simply relates the two chemical potentials

$$\tilde\mu = \mu + \frac{(\lambda \pi \rho_0)^2}{2m}\,, \tag{B.11}$$

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setting no further conditions on $\rho(x)$, where ρ_0 is the subtraction constant associated with the ρ in question, and should not be confused with the one appearing in (49).

To summarize - any solution of (47) with chemical potential μ is simultaneously a solutions of (B.3) with chemical potential $\tilde{\mu}$ given by (B.11).

Appendix C: A Brief Summary of the Collective Field Formulation of the Calogero Model

In order for this paper to be self-contained, we briefly summarize in this appendix the derivation of the collective-field Hamiltonian (2) from (1) .

The singularities of the Calogero-model Hamiltonian (1), namely,

$$H = -\frac{1}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\lambda(\lambda-1)}{2m} \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2} \, , \tag{C.1}$$

at points where particles coincide, implies that the many-body eigenfunctions contain a Jastrow-type prefactor

$$\Pi = \prod_{i<j}^{N} (x_i - x_j)^\lambda \, . \tag{C.2}$$

This Jastrow factor vanishes (for positive λ) at particle coincidence points, and multiplies that part of the wave-function which is totally symmetric under any permutation of particles[7]. It is precisely these symmetric wave-functions on which the collective field operators act, as explained below.

Let us recall at this point some of the basic ideas of the collective-field method [8, 9, 10], adapted specifically to the Calogero model[11, 12]: Instead of solving the Schrödinger equation associated with (C.1) for the many-body eigenfunctions, subjected to the appropriate particle statistics (Bosonic, Fermionic of fractional), we restrict ourselves to functions which are totally symmetric under any permutation of identical particles. This we achieve by stripping off the Jastrow factor (C.2) from

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[7]Note, in particular, that for $\lambda = 0$ and $\lambda = 1$, the model describes interacting bosons and fermions, respectively.

the eigenfunctions, which means performing on (C.1) the similarity transformation

$$H \to \tilde{H} = \Pi^{-1} H \Pi \,, \tag{C.3}$$

where the Hamiltonian

$$\tilde{H} = -\frac{1}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i{}^2} - \frac{\lambda}{m} \sum_{i \neq j}^{N} \frac{1}{x_i - x_j} \frac{\partial}{\partial x_i} \,. \tag{C.4}$$

Note that $\tilde{H}$ does not contain the singular two-body interactions. By construction, this Hamiltonian is hermitian with respect to the measure

$$d\mu(x_i) = \Pi^2 \, d^N x \,,$$

(as opposed to the original Hamiltonian H in (C.1), which is hermitian with respect to the flat Cartesian measure).

We can think of the symmetric many-body wave-functions acted upon by $\tilde{H}$ as functions depending on all possible symmetric combinations of particle coordinates. These combinations form an overcomplete set of variables. However, as explained below, in the *continuum* limit, redundancy of these symmetric variables has a negligible effect. The set of these symmetric variables can be generated, for example, by producs of moments of the collective - or density - field

$$\rho(x) = \sum_{i=1}^{N} \delta(x - x_i) \,. \tag{C.5}$$

The collective-field theory for the Calogero model is obtained by changing variables from the particle coordinates x_i to the density field $\rho(x)$. This transformation replaces the finitely many variables x_i by a continuous field, which is just another manifestation of overcompleteness of the collective variables. Clearly, description of the particle systems in terms of continuous fields becomes an effectively good

description in the high density limit. Of course, the large density limit means that we have taken the large- N limit.

Changing variables from particle coordinates x_i to the collective fields (C.5) implies that we should express all partial derivatives in the Hamiltonian $\tilde{H}$ in (C.4) as

$$\frac{\partial}{\partial x_i} = \int dx \frac{\partial \rho(x)}{\partial x_i} \frac{\delta}{\delta \rho(x)} , \tag{C.6}$$

where we applied the differentiation chain rule.

In the large $-N$ limit, the Hamiltonian $\tilde{H}$ can be expressed entirely in terms of the collective field $\rho(x)$ and its canonical conjugate momentum

$$\pi(x) = -i \frac{\delta}{\delta \rho(x)} , \tag{C.7}$$

as we show below. It follows from (C.6) and (C.7) that the particle momentum operators (acting on symmetric wave-functions) may be expressed in terms of the collective-field momenta at particular points on the line as

$$p_i = -\pi'(x_i) \tag{C.8}$$

(where $\pi'(x) = \partial_x \pi(x)$). Finally, note from (C.5) that the collective field obeys the normalization condition

$$\int dx \rho(x) = N . \tag{C.9}$$

The density field ρ and its conjugate momentum π satisfy the equal-time canonical commutation relations[8]

$$[\rho(x), \pi(y)] = i\delta(x - y) , \tag{C.10}$$

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[8]According to (C.9), the zero-momentum modes of the density fields are constrained, i.e., non-dynamical. This affects the commutation relation (C.10), whose precise form is $[\rho(x), \pi(y)] = i(\delta(x - y) - (1/l))$, where l is the size of the large one-dimensional box in which the system is quantized, which is much larger than the macroscopic size L of the particle condensate in the system. In what follows, we can safely ignore this $1/l$ correction in the commutation relations.

(and of course $[\rho(x), \rho(y)] = [\pi(x), \pi(y)] = 0)$. By substituting (C.5)-(C.8) in (C.4), we obtain the continuum-limit expression for $\tilde{H}$ as

$$\tilde{H} = \frac{1}{2m} \int dx \rho(x)(\partial_x \pi(x))^2 - \frac{i}{m} \int dx \rho(x) \left(\frac{\lambda - 1}{2} \frac{\partial_x \rho}{\rho} + \lambda \fint \frac{dy \rho(y)}{x - y} \right) \partial_x \pi(x) \tag{C.11}$$

where $\fint$ denotes Cauchy's principal value.

It can be shown [8] that (C.11) is hermitian with respect to the functional measure[9]

$$\mathcal{D}\mu[\rho] = J[\rho] \prod_x d\rho(x), \tag{C.12}$$

where $J[\rho]$ is the Jacobian of the transformation from the $\{x_i\}$ to the collective field $\{\rho(x)\}$. In the large - N limit it is given by [32]

$$\ln J = (1 - \lambda) \int dx \rho(x) \ln \rho(x) - \lambda \int dx dy \rho(x) \ln |x - y| \rho(y) \tag{C.13}$$

It is more convenient to work with a Hamiltonian, which unlike (C.11), is hermitian with respect to the flat functional Cartesian measure $\prod_x d\rho(x)$. This we achieve by means of the similarity transformation $\psi \to J^{\frac{1}{2}}\psi$, $\tilde{H} \to H_{coll} = J^{\frac{1}{2}}\tilde{H}J^{-\frac{1}{2}}$, where the continuum *collective* Hamiltonian is

$$H_{coll} = \frac{1}{2m} \int dx\, \pi'(x)\, \rho(x)\, \pi'(x) + \frac{1}{2m} \int dx \rho(x) \left(\frac{\lambda - 1}{2} \frac{\partial_x \rho}{\rho} + \lambda \fint \frac{dy \rho(y)}{x - y} \right)^2 + H_{sing}, \tag{C.14}$$

namely, the Hamiltonian given by (2) and (3). The collective-field Hamiltonian (12) of the two-family Calogero model can be derived from (9) in a similar manner.

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[9]By definition (recall (C.5)), this measure is defined only over positive values of ρ.

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